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Prediction of viscoelastic property of layered materials

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Abstract

In this paper, a homogenization-based method for predicting the viscoelastic properties of layered materials is presented, and the explicit formulae for predicting the viscoelastic relaxation modulus of layered materials are obtained. The derivation process includes two steps. First, the Laplace transform is applied to the governing equation of the viscoelastic problem of layered materials, and the Laplace-transformed effective relaxation modulus of layered materials is derived analytically based on the homogenization theory. Second, the effective relaxation modulus in the time domain is obtained using the inverse Laplace transformation. A numerical example is presented at the end of the paper.

Keywords: Layered materials; Viscoelasticity; Homogenization; Laplace transformation

1. Introduction

Generally speaking, most natural and artificial materials are heterogeneous in a microscopic scale. A typical kind of heterogeneous material is composite material, which may be defined as a man-made material with dissimilar constituents which occupy different regions with distinct interfaces between them (Kalamkarov, 1992). Owing to the wide application of composite materials in high performance structures, the property analysis of heterogeneous materials becomes more and more important. Unfortunately, it is extremely difficult to determine the responses of the structures consisting of such materials with a large number of heterogeneities. One way to overcome this difficulty is to replace the heterogeneous composite material with an equivalent homogeneous material, which can represent both the composite material's effective properties and the influence features of their heterogeneity in microscopic scale. Although it is, in principle, possible to determine the equivalent material properties experimentally, it is, in practice, very costly and unrealistic to carry out such experiments for all possible microstructures.

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Homogenization theory, a mathematical theory originating from the study of partial differential equations with rapidly varying coefficients, is an alternative approach to determine the effective properties of composite materials (Bensoussan et al., 1978; Sanchez-Palencia, 1980). The homogenization method assumes that all quantities vary in two scales, i.e., a local scale and a global scale. Due to the periodicity of the microstructure, quantities, such as displacement, strain, and stress, are assumed to be periodic with respect to the local scale. In order to find the effective material properties of a medium, the asymptotic behavior of the medium as the period goes to zero is investigated. Mathematically, homogenization theory is a limit theory which uses the asymptotic expansions and the assumption of periodicity to substitute the differential equations with rapidly oscillating coefficients with differential equations whose coefficients are constant or slowly varying in such a way that the solutions are close to the initial equations (Oleinik, 1984). At this point, it should be noted that the homogenization method has a rigorous mathematical background. Besides, it is readily implemented with finite element method and thus especially useful for microstructures with complex and irregular configurations (Guedes and Kikuchi, 1990). Owing to these attractive features, the homogenization method was widely used in the past few years in prediction of elastic constants, thermoelastic properties and thermal conductivity, and in topology optimization of structures, among others, referred to in the survey papers by Hassani and Hinton (1998a,b).

There have been some publications (e.g., Yi et al., 1998; Nguyen et al., 1995) on the applications of homogenization theory to viscoelastic problems, although the number is small compared with that in the elastic cases. Yi et al. (1998) presented a systematic way of obtaining the effective viscoelastic modulus in the time domain. This method requires implementing a numerical inverse Laplace transformation, which is not always easy. In this paper, the viscoelastic properties of layered materials are investigated. For layered materials, the elastic homogenization problem can be solved by an analytical method (Hassani and Hinton, 1998a,b), and the viscoelastic homogenization problem in transformed space is similar to the elastic one. Based on these observations, the explicit formulae for predicting the effective viscoelastic relaxation modulus of layered materials are derived by using homogenization theory in transformed space and inverse Laplace transformation.

2. Viscoelastic problem and Laplace transformation of layered materials

The composite material investigated herein has a periodic layered microstructure as shown in Fig. 1. Each layer consists of a homogeneous material. Based on the homogenization theory (Bensoussan et al., 1978; Sanchez-Palencia, 1980; Yi et al., 1998), the viscoelastic problem of layered materials can be formulated in a macroscopic or global coordinate system $\mathbf{x} = (x_1, x_2, x_3)^{\mathrm{T}}$ and a microscopic or local coordinate system $y = x_3/\varepsilon$. Where, ε is a small positive parameter which is the ratio of the microscopic and macroscopic dimensions. Material properties depend on the microscopic variable, while variables such as

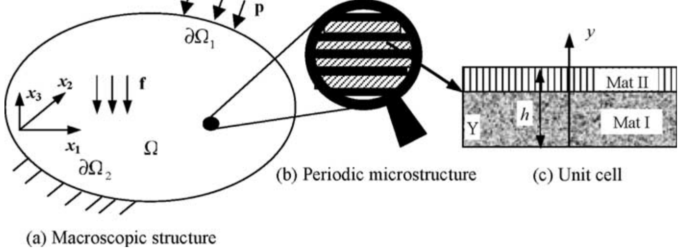

Fig. 1. Periodic microstructure and unit cell of layered material body.

displacements and strain/stress fields depend on both the microscopic and macroscopic variables. All these quantities mentioned above are the functions of two coordinate systems: $g^{\varepsilon}(\mathbf{x}) = g(\mathbf{x}, y)$. Owing to the periodic character of the microstructure, the dependency of a function on y is Y-periodic and the function can be expressed as:

$$g^{\varepsilon}(\mathbf{x}) = g(\mathbf{x}, y) = g(\mathbf{x}, y + \mathbf{Y}) \tag{1}$$

It should be noted that the periodic feature is only exhibited in the normal direction of the layers. In this case, the derivatives of a general function with respect to coordinates x_i $(i = 1, 2, 3)$ should be formulated as

$$\frac{\partial g^{\varepsilon}(\mathbf{x})}{\partial x_i} = \begin{cases} \frac{\partial g(\mathbf{x},y)}{\partial x_i}, & i = 1, 2 \\ \frac{\partial g(\mathbf{x},y)}{\partial x_i} + \frac{1}{\varepsilon}\frac{\partial g(\mathbf{x},y)}{\partial y}, & i = 3 \end{cases} \tag{2}$$

Let Ω, an open connected domain of R^3, be the domain occupied by the layered material, $\partial\Omega_1$ and $\partial\Omega_2$ be its outer boundaries with specific surface traction and specific displacements respectively, see Fig. 1. The unit cell is expressed by $Y = [0, h]$ which is a one-dimensional region in the layers' normal direction.

Define strain operators $\varepsilon_x(\bullet)$ and $\varepsilon_y(\bullet)$ as follows

$$\varepsilon_x(\bullet) = \begin{bmatrix} \frac{\partial}{\partial x_1} & 0 & 0 & 0 & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ 0 & \frac{\partial}{\partial x_2} & 0 & \frac{\partial}{\partial x_3} & 0 & \frac{\partial}{\partial x_1} \\ 0 & 0 & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 \end{bmatrix}^{\mathrm{T}} \tag{3a}$$

$$\varepsilon_y(\bullet) = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & 0 & \frac{\partial}{\partial y} & 0 & 0 \\ 0 & 0 & \frac{\partial}{\partial y} & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}} \tag{3b}$$

Then, the constitutive equation of viscoelastic problems can be written as (Christensen, 1982)

$$\{\sigma^{\varepsilon}(\mathbf{x}, t)\} = \int_0^t \mathbf{G}(\mathbf{x}, t - \tau)\frac{\partial}{\partial t}(\varepsilon_x(\mathbf{u}^{\varepsilon}(\mathbf{x}, \tau)))\,\mathrm{d}\tau \tag{4}$$

where t denotes the time, $\mathbf{G}$ is the relaxation modulus matrix, and $\{\sigma^{\varepsilon}(\mathbf{x}, t)\}$ and $\mathbf{u}^{\varepsilon}(\mathbf{x}, t)$ are stress and displacement vectors respectively.

Based on the virtual work principle, the viscoelastic governing equation can be constructed as:

Find $\mathbf{u}^{\varepsilon}(\mathbf{x}, t) \in \mathbf{V}^{\varepsilon}$, such that

$$\int_{\Omega} \varepsilon_x^{\mathrm{T}}(\delta\mathbf{u}(\mathbf{x}))\sigma^{\varepsilon}(\mathbf{x}, t)\,\mathrm{d}\mathbf{x} - \int_{\Omega} \delta\mathbf{u}^{\mathrm{T}}\mathbf{f}\,\mathrm{d}\mathbf{x} - \int_{\partial\Omega_1} \delta\mathbf{u}^{\mathrm{T}}\mathbf{p}\,\mathrm{d}\mathbf{x} = 0, \quad \forall\delta\mathbf{u} \in \mathbf{V}^{\varepsilon} \tag{5}$$

where, $\mathbf{f}$ and $\mathbf{p}$ are body forces and surface traction, respectively. The set $\mathbf{V}^{\varepsilon}$ including the kinetic admissible displacements is defined by

$$\mathbf{V}^{\varepsilon} = \{\mathbf{u}(\mathbf{x}, t)|_{t=t_0} \in (H^1(\Omega))^3 \text{and } \mathbf{u}|_{\partial\Omega_2} = 0\} \tag{6}$$

and $H^1(\Omega)$ is the Sobolev space.

Applying Laplace transformation to Eq. (5) yields

$$\int_{\Omega} \varepsilon_x^{\mathrm{T}}(\delta\mathbf{u}(\mathbf{x}))s\widetilde{\mathbf{G}}(\mathbf{x}, s)\varepsilon_x(\tilde{\mathbf{u}}^{\varepsilon}(\mathbf{x}, s))\,\mathrm{d}\mathbf{x} - \int_{\Omega} \delta\mathbf{u}^{\mathrm{T}}\tilde{\mathbf{f}}\,\mathrm{d}\mathbf{x} - \int_{\partial\Omega_1} \delta\mathbf{u}^{\mathrm{T}}\tilde{\mathbf{p}}\,\mathrm{d}\mathbf{x} = 0, \quad \forall\delta\mathbf{u} \in \mathbf{V}^{\varepsilon} \tag{7}$$

where variables with the mark "$\sim$" show that they are Laplace transformed, and s is the transformed parameter. For example, if $f(t)$ is a general function, $\tilde{f}(s)$ denotes its Laplace transformation

$$\tilde{f}(s) = \int_0^{\infty} f(t)\mathrm{e}^{-st}\,\mathrm{d}t \tag{8}$$

Following the homogenization procedure (Bensoussan et al., 1978), we expand the displacement $\mathbf{u}^{\varepsilon}(\mathbf{x}, t)$ into an asymptotic series in the following form

$$\mathbf{u}^{\varepsilon}(\mathbf{x}, t) = \mathbf{u}^0(\mathbf{x}, t) + \varepsilon\mathbf{u}^1(\mathbf{x}, \mathbf{y}, t) + \varepsilon^2\mathbf{u}^2(\mathbf{x}, \mathbf{y}, t) + \cdots \tag{9}$$

where $\mathbf{u}^0(\mathbf{x}, t)$ is effective or overall displacement, only depending on the macroscopic coordinates $\mathbf{x}$ (Kalamkarov, 1992). Substituting the Laplace transformation of Eq. (9) into (7), and equating the terms with the same power of ε, the following expressions can be obtained

$$\int_{\Omega} s\varepsilon_y^{\mathrm{T}}(\delta\mathbf{u})\widetilde{\mathbf{G}}[\varepsilon_x(\tilde{\mathbf{u}}^0) + \varepsilon_y(\tilde{\mathbf{u}}^1)]\,\mathrm{d}\mathbf{x} = 0, \quad \forall\delta\mathbf{u} \in \mathbf{V}_{\Omega\times Y} \tag{10}$$

$$\int_{\Omega}[\varepsilon_y^{\mathrm{T}}(\delta\mathbf{u})\widetilde{\mathbf{G}}(\varepsilon_x(\tilde{\mathbf{u}}^1) + \varepsilon_y(\tilde{\mathbf{u}}^2)) + \varepsilon_x^{\mathrm{T}}(\delta\mathbf{u})\mathbf{C}(\varepsilon_x(\tilde{\mathbf{u}}^0) + \varepsilon_y(\tilde{\mathbf{u}}^1))]\,\mathrm{d}\mathbf{x}$$
$$- \int_A (\delta\mathbf{u})^{\mathrm{T}}\tilde{\mathbf{f}}\,\mathrm{d}\mathbf{x} - \int_{\partial\Omega_1}(\delta\mathbf{u})^{\mathrm{T}}\tilde{\mathbf{p}}\,\mathrm{d}\mathbf{x} = 0, \quad \forall\delta\mathbf{u} \in \mathbf{V}_{\Omega\times Y} \tag{11}$$

where

$$\mathbf{V}_{\Omega\times Y} = \{\mathbf{u}(\mathbf{x}, y); (\mathbf{x}, y) \in \Omega \times Y \,|\, \mathbf{u}(\mathbf{x}, y) = \mathbf{u}(\mathbf{x}, y + Y), \mathbf{u} \text{ smooth enough and } \mathbf{u}|_{\partial\Omega_2} = 0\} \tag{12}$$

Similarly, we define $\mathbf{V}_{\Omega}$ and $\mathbf{V}_Y$ as

$$\mathbf{V}_{\Omega} = \{\mathbf{u}(\mathbf{x}); \mathbf{x} \in \Omega \,|\, \mathbf{u}(\mathbf{x}) \in H^1(\Omega) \text{ and } \mathbf{u}(\mathbf{x})|_{\partial\Omega_2} = 0\} \tag{13}$$

$$\mathbf{V}_Y = \{\mathbf{u}(y); y \in Y|\mathbf{u}(y) \in H^1(\Omega), \mathbf{u}(y) = \mathbf{u}(y + Y)\} \tag{14}$$

For a Y-periodic function, we have (Bensoussan et al., 1978; Sanchez-Palencia, 1980)

$$\lim_{\varepsilon\to 0^+} \int_{\Omega} g^{\varepsilon}(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_{\Omega}\left\{\frac{1}{|Y|}\int_Y g(\mathbf{x}, y)\,\mathrm{d}y\right\}\mathrm{d}\mathbf{x} \tag{15}$$

Considering the above equation, we know that when $\varepsilon \to 0^+$, Eqs. (10) and (11) become

$$\int_{\Omega}\left\{\frac{1}{|Y|}\int_Y s\varepsilon_y^{\mathrm{T}}(\delta\mathbf{u})\widetilde{\mathbf{G}}(\varepsilon_x(\tilde{\mathbf{u}}^0) + \varepsilon_y(\tilde{\mathbf{u}}^1))\,\mathrm{d}y\right\}\mathrm{d}\mathbf{x} = 0, \quad \forall\delta\mathbf{u} \in \mathbf{V}_{\Omega\times Y} \tag{16}$$

$$\int_{\Omega}\left\{\frac{1}{|Y|}\int_Y[\varepsilon_y^{\mathrm{T}}(\delta\mathbf{u})\widetilde{\mathbf{G}}(\varepsilon_x(\tilde{\mathbf{u}}^1) + \varepsilon_y(\tilde{\mathbf{u}}^2)) + \varepsilon_x^{\mathrm{T}}(\delta\mathbf{u})\mathbf{C}(\varepsilon_x(\tilde{\mathbf{u}}^0) + \varepsilon_y(\tilde{\mathbf{u}}^1))]\,\mathrm{d}y\right\}\mathrm{d}\mathbf{x}$$
$$- \int_{\Omega}\left\{\frac{1}{|Y|}\int_Y(\delta\mathbf{u})^{\mathrm{T}}\tilde{\mathbf{f}}\,\mathrm{d}y\right\}\mathrm{d}\mathbf{x} - \int_{\partial\Omega_1}(\delta\mathbf{u})^{\mathrm{T}}\tilde{\mathbf{p}}\,\mathrm{d}\mathbf{x} = 0, \quad \forall\delta\mathbf{u} \in \mathbf{V}_{\Omega\times Y} \tag{17}$$

As $\delta\mathbf{u}$ is an arbitrary function we choose $\delta\mathbf{u} = \delta\mathbf{u}(y)$. Eq. (16) becomes

$$\int_Y \varepsilon_y^{\mathrm{T}}(\delta\mathbf{u})\widetilde{\mathbf{G}}(\varepsilon_x(\tilde{\mathbf{u}}^0) + \varepsilon_y(\tilde{\mathbf{u}}^1))\,\mathrm{d}y = 0, \quad \forall\delta\mathbf{u} \in \mathbf{V}_Y \tag{18}$$

Although this equation does not have a unique solution, its solution can be determined up to an additive constant (Sanchez-Palencia, 1980). As this equation is linear with respect to $\tilde{\mathbf{u}}^0$, its solution $\tilde{\mathbf{u}}^1$ can be expressed in terms of $\tilde{\mathbf{u}}^0$ as:

$$\tilde{\mathbf{u}}^1(\mathbf{x}, y, s) = -\Phi(y, s)\varepsilon_x(\tilde{\mathbf{u}}^0(\mathbf{x}, s)) + \Psi(\mathbf{x}) \tag{19}$$

where $\Psi(\mathbf{x})$ is arbitrary constant of integration in y. $\Phi(y,s)$ has the form

$$\Phi(y,s) = [\Phi^1(y,s), \Phi^2(y,s), \ldots, \Phi^6(y,s)], \quad \Phi^j(y,s) = (\Phi_1^j, \Phi_2^j, \Phi_3^j)^{\mathrm{T}}, \quad j = 1,2,\ldots,6 \tag{20}$$

and is the periodic solution of the following microscopic homogenization problem

$$s\int_Y \varepsilon_y^{\mathrm{T}}(\delta\mathbf{u}(\mathbf{y}))[\widetilde{\mathbf{G}}^j - \widetilde{\mathbf{G}}\varepsilon_y(\Phi^j(y,s))]\,\mathrm{d}y = 0, \quad \forall\delta\mathbf{u} \in \mathbf{V}_Y, \quad j = 1,2,\ldots,6 \tag{21}$$

In the above equation, $\widetilde{\mathbf{G}}^j$ is the jth column vector of the Laplace transformation of the relaxation modulus matrix.

$$\widetilde{\mathbf{G}}^j = (G_{1j}, G_{2j}, \ldots, G_{6j})^{\mathrm{T}} \tag{22}$$

Substituting Eq. (19) into Eq. (17) and choosing $\delta\mathbf{u} = \delta\mathbf{u}(\mathbf{x})$ yields

$$\int_\Omega s\varepsilon_x^{\mathrm{T}}(\delta\mathbf{u})\widetilde{\mathbf{G}}^H\varepsilon_x(\tilde{\mathbf{u}}^0)\,\mathrm{d}\mathbf{x} - \int_Z \delta\mathbf{u}^{\mathrm{T}}\tilde{\mathbf{f}}^H\,\mathrm{d}\mathbf{x} - \int_{\partial\Omega_1}\delta\mathbf{u}^{\mathrm{T}}\tilde{\mathbf{p}}\,\mathrm{d}\Gamma = 0, \quad \forall\delta\mathbf{u} \in \mathbf{V}_\Omega \tag{23}$$

where $\tilde{\mathbf{f}}^H = \frac{1}{|Y|}\int_Y \tilde{\mathbf{f}}\,\mathrm{d}y$ and

$$\widetilde{\mathbf{G}}^H(s) = \frac{1}{|\mathbf{Y}|}\int_Y [\widetilde{\mathbf{G}} - \widetilde{\mathbf{G}}\varepsilon_y(\Phi(y,s))]\,\mathrm{d}y \tag{24}$$

$$\varepsilon_y(\Phi(y,s)) = [\varepsilon_y(\Phi^1(y,s)), \varepsilon_y(\Phi^2(y,s)), \ldots, \varepsilon_y(\Phi^6(y,s))] \tag{25}$$

Eq. (23) is very similar to the governing Eq. (7) of virtual work principle. $\widetilde{\mathbf{G}}^H$ is the effective relaxation modulus matrix in the Laplace transformed domain. After obtaining $\widetilde{\mathbf{G}}^H$ from Eq. (24), the effective relaxation modulus $\mathbf{G}^H$ of the layered material in the time domain can be determined by the inverse Laplace transformation.

Eq. (21) is called a microscopic homogenization problem. In general cases, it is solved by a numerical method, such as the finite element method. For layered material investigated in this paper, it will be solved analytically.

3. Solution of microscopic homogenization problem

Assume that the material of every layer is orthotropic and the plane parallel to the layer is a material symmetric plane. In this case, the relaxation modulus of every layer has the following form:

$$\mathbf{G} = \begin{bmatrix} G_{11} & G_{12} & G_{13} & 0 & 0 & 0 \\ G_{12} & G_{22} & G_{23} & 0 & 0 & 0 \\ G_{13} & G_{23} & G_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & G_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & G_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & G_{66} \end{bmatrix} \tag{26}$$

Substituting Eq. (26) into Eq. (21) and considering the fact that Eq. (21) is defined in one-dimensional region, we have

$$\int_Y (\varepsilon_y^{\mathrm{T}}(\delta\mathbf{u})(\widetilde{\mathbf{G}}^j - \widetilde{\mathbf{G}}\varepsilon_y(\Phi^j))\,\mathrm{d}y = 0 \tag{27}$$

Introducing Eq. (3b) into Eq. (27) and expanding this equation yields

$$\int_Y \left(\widetilde{G}_{3j} - \widetilde{G}_{33}\frac{\partial \Phi_{3j}}{\partial y}\right)\frac{\partial \delta u_3}{\partial y}\,\mathrm{d}y = 0, \quad j = 1,2,\ldots,6 \tag{28a}$$

$$\int_Y \left(\widetilde{G}_{4j} - \widetilde{G}_{44}\frac{\partial \Phi_{2j}}{\partial y}\right)\frac{\partial \delta u_2}{\partial y}\,\mathrm{d}y = 0, \quad j = 1,2,\ldots,6 \tag{28b}$$

$$\int_Y \left(\widetilde{G}_{5j} - \widetilde{G}_{55}\frac{\partial \Phi_{1j}}{\partial y}\right)\frac{\partial \delta u_1}{\partial y}\,\mathrm{d}y = 0, \quad j = 1,2,\ldots,6 \tag{28c}$$

Integrating Eq. (28) by parts yields

$$-\frac{\partial \Phi_{3j}}{\partial y} = \frac{C_{3j}}{\widetilde{G}_{33}} - \frac{\widetilde{G}_{3j}}{\widetilde{G}_{33}}, \quad j = 1,2,\ldots,6 \tag{29a}$$

$$-\frac{\partial \Phi_{1j}}{\partial y} = \frac{C_{1j} - \widetilde{G}_{5j}}{\widetilde{G}_{55}}, \quad j = 1,2,\ldots,6 \tag{29b}$$

$$-\frac{\partial \Phi_{2j}}{\partial y} = \frac{C_{2j} - \widetilde{G}_{4j}}{\widetilde{G}_{44}}, \quad j = 1,2,\ldots,6 \tag{29c}$$

where C_{1j}, C_{2j} and C_{3j} $(j = 1,2,\ldots,6)$ are real constants.

The periodicity condition requires that the following equations should be satisfied

$$\frac{1}{|Y|}\int_Y \frac{\partial \Phi_{3j}}{\partial y}\,\mathrm{d}y = 0, \quad j = 1,2,\ldots,6 \tag{30a}$$

$$\frac{1}{|Y|}\int_Y \frac{\partial \Phi_{1j}}{\partial y}\,\mathrm{d}y = 0, \quad j = 1,2,\ldots,6 \tag{30b}$$

$$\frac{1}{|Y|}\int_Y \frac{\partial \Phi_{2j}}{\partial y}\,\mathrm{d}y = 0, \quad j = 1,2,\ldots,6 \tag{30c}$$

From the above equations, the non-zero components of the real constants C_{1j}, C_{2j} and C_{3j} $(j = 1,2,\ldots,6)$ can be obtained by

$$C_{15} = \frac{1}{M(1/\widetilde{G}_{55})}, \quad C_{24} = \frac{1}{M(1/\widetilde{G}_{44})}, \quad C_{3j} = \frac{M(\widetilde{G}_{3j}/\widetilde{G}_{33})}{M(1/\widetilde{G}_{33})}, \quad j = 1,2,3 \tag{31}$$

Substituting Eqs. (29) and (31) into Eq. (24), the Laplace transformed effective relaxation moduli can be expressed explicitly as

$$\widetilde{G}^H_{33} = 1/M(1/\widetilde{G}_{33}) \tag{32a}$$

$$\widetilde{G}^H_{11} = M(\widetilde{G}_{11}) - M(\widetilde{G}^2_{13}/\widetilde{G}_{33}) + \widetilde{G}^H_{33}M^2(\widetilde{G}_{13}/\widetilde{G}_{33}) \tag{32b}$$

$$\widetilde{G}^H_{22} = M(\widetilde{G}_{22}) - M(\widetilde{G}^2_{23}/\widetilde{G}_{33}) + \widetilde{G}^H_{33}M^2(\widetilde{G}_{23}/\widetilde{G}_{33}) \tag{32c}$$

$$\widetilde{G}^H_{12} = M(\widetilde{G}_{12}) - M(\widetilde{G}_{23}\widetilde{G}_{13}/\widetilde{G}_{33}) + \widetilde{G}^H_{33}M(\widetilde{G}_{13}/\widetilde{G}_{33})M(\widetilde{G}_{32}/\widetilde{G}_{33}) \tag{32d}$$

$$\widetilde{G}_{13}^{H} = \widetilde{G}_{33}^{H} M(\widetilde{G}_{31}/\widetilde{G}_{33}) \tag{32e}$$

$$\widetilde{G}_{23}^{H} = \widetilde{G}_{33}^{H} M(\widetilde{G}_{23}/\widetilde{G}_{33}) \tag{32f}$$

$$\widetilde{G}_{44}^{H} = 1/M(1/\widetilde{G}_{44}) \tag{32g}$$

$$\widetilde{G}_{55}^{H} = 1/M(1/\widetilde{G}_{55}) \tag{32h}$$

$$\widetilde{G}_{66}^{H} = M(\widetilde{G}_{66}) \tag{32i}$$

where the volume average operator is defined as

$$M(*) = \frac{1}{|Y|}\int_{Y} (*)\,\mathrm{d}y = \frac{1}{h}\int_{0}^{h} (*)\,\mathrm{d}y \tag{33}$$

4. Inverse Laplace transformation of relaxation modulus

In this section, two typical kinds of viscoelastic constitutive models will be considered in order to illustrate the procedure of the method.

4.1. Model I

Assume that the viscoelastic properties of every layer satisfy the following three-parameter model

$$G_{ij}(t) = E(t)\overline{G}_{ij} = (q_0 + q\mathrm{e}^{-pt})\overline{G}_{ij} \tag{34}$$

where $\overline{G}_{ij}$, q_0, q and p are constant parameters. The material constants of two different layers are expressed as

$$E(t) = E^{\mathrm{I}}(t) = q_0^{\mathrm{I}} + q^{\mathrm{I}}\mathrm{e}^{-p^{\mathrm{I}}}, \quad \overline{G}_{ij} = \overline{G}_{ij}^{\mathrm{I}} \tag{35a}$$

$$E(t) = E^{\mathrm{II}}(t) = q_0^{\mathrm{II}} + q^{\mathrm{II}}\mathrm{e}^{-p^{\mathrm{II}}t}, \quad \overline{G}_{ij} = \overline{G}_{ij}^{\mathrm{II}} \tag{35b}$$

In the above equations and the next sections, the superscripts I and II denote different layers I and II. The Laplace transformed relaxation modulus of two different layers are expressed as

$$\widetilde{G}_{ij}^{\mathrm{I}}(s) = \widetilde{E}^{\mathrm{I}}(s)\overline{G}_{ij}^{\mathrm{I}} = \left(\frac{q_0^{\mathrm{I}}}{s} + \frac{q^{\mathrm{I}}}{s+p^{\mathrm{I}}}\right)\overline{G}_{ij}^{\mathrm{I}} = \frac{(q_0^{\mathrm{I}} + q^{\mathrm{I}})s + p^{\mathrm{I}}q_0^{\mathrm{I}}}{s(s+p^{\mathrm{I}})}\overline{G}_{ij}^{\mathrm{I}} \tag{36a}$$

$$\widetilde{G}_{ij}^{\mathrm{II}}(s) = \widetilde{E}^{\mathrm{II}}(s)\overline{G}_{ij}^{\mathrm{II}} = \left(\frac{q_0^{\mathrm{II}}}{s} + \frac{q^{\mathrm{II}}}{s+p^{\mathrm{II}}}\right)\overline{G}_{ij}^{\mathrm{II}} = \frac{(q_0^{\mathrm{II}} + q^{\mathrm{II}})s + p^{\mathrm{II}}q_0^{\mathrm{II}}}{s(s+p^{\mathrm{II}})}\overline{G}_{ij}^{\mathrm{II}} \tag{36b}$$

Introducing $\widetilde{G}_{ij}^{\mathrm{I}}(s)$ and $\widetilde{G}_{ij}^{\mathrm{II}}(s)$ into Eq. (32) and doing inverse transformation, we have

$$G_{33}^{H} = \mathrm{Rev}(1/M(1/\widetilde{G}_{33})) \tag{37a}$$

$$G_{11}^{H} = M(G_{11}) - M(E\overline{G}_{13}^{2}/\overline{G}_{33}) + G_{33}^{H}M^{2}(\overline{G}_{13}/\overline{G}_{33}) \tag{37b}$$

$$G_{22}^{H} = M(G_{22}) - M(E\overline{G}_{23}^{2}/\overline{G}_{33}) + G_{33}^{H}M^{2}(\overline{G}_{23}/\overline{G}_{33}) \tag{37c}$$

$$G_{12}^H = M(G_{12}) - M(E\overline{G}_{23}\overline{G}_{13}/\overline{G}_{33}) + M(\overline{G}_{13}/\overline{G}_{33})M(\overline{G}_{32}/\overline{G}_{33})G_{33}^H \tag{37d}$$

$$G_{13}^H = M(\overline{G}_{31}/\overline{G}_{33})G_{33}^H \tag{37e}$$

$$G_{23}^H = M(\overline{G}_{23}/\overline{G}_{33})G_{33}^H \tag{37f}$$

$$G_{44}^H = \mathrm{Rev}(1/M(1/\widetilde{G}_{44})) \tag{37g}$$

$$G_{55}^H = \mathrm{Rev}(1/M(1/\widetilde{G}_{55})) \tag{37h}$$

$$G_{66}^H = M(G_{66}) \tag{37i}$$

where Rev( ) denotes inverse Laplace transformation. In the above equations, all the elements of the relaxation modulus except G_{33}^H, G_{44}^H and G_{55}^H are given explicitly in terms of the properties of the layers. Next, we shall determine them by the inverse Lalace transformation of $(1/M(1/\widetilde{G}_{ii})$, $i = 3,4,5$.

In this paper, we will use the Cauchy residue theorem based method to determine the inverse Laplace transformation. The inverse transform of a general function $\tilde{f}(s)$ is defined by the complex integral formula (Debnath, 1995)

$$f(t) = \mathrm{Rev}(\tilde{f}(s)) = \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \tilde{f}(s)\mathrm{e}^{st}\,\mathrm{d}s \tag{38}$$

where c is a suitable real constant and $\tilde{f}(s)$ is an analytic function of the complex variable s in the right half-plane $\mathrm{Re}\, s > c$. Suppose that $\tilde{f}(s)$ is a single valued function with a finite or enumerably infinite number of polar singularities and that all the singularities lie in the left half-plane Re $s \leqslant c$. Then, Cauchy residue theorem (Debnath, 1995) yields

$$f(t) = \mathrm{Rev}(\tilde{f}(s)) = \text{ sum of the residues of } \tilde{f}(s)\ \mathrm{e}^{st} \text{ at the poles of } \tilde{f}(s) \tag{39}$$

Considering the definition of the volume average operator $M(*)$ given by Eq. (33), $1/M(1/\widetilde{G}_{ii})$, $(i = 3,4,5)$, can be expressed as

$$1/M(1/\widetilde{G}_{ii}) = \frac{1}{V^{\mathrm{I}}/\widetilde{G}_{ii}^{\mathrm{I}} + V^{\mathrm{II}}/\widetilde{G}_{ii}^{\mathrm{II}}} = \frac{P_{ii}(s)}{s(s-s_1^{ii})(s-s_2^{ii})}, \quad i = 3,4,5 \tag{40}$$

where V^{I} and V^{II} are the volume fractions of MAT-I layer and MAT-II layer, respectively, $P_{ii}(s)$ is a polynomial of state variable s and is defined by the following equations

$$P_{ii}(s) = \overline{G}_{ii}^{\mathrm{I}}\overline{G}_{ii}^{\mathrm{II}}(A^{\mathrm{I}}s + p^{\mathrm{I}}q_0^{\mathrm{I}})(A^{\mathrm{II}}s + p^{\mathrm{II}}q_0^{\mathrm{II}}) \tag{41}$$

$$A^{\mathrm{I}} = q_0^{\mathrm{I}} + q^{\mathrm{I}}, \quad A^{\mathrm{II}} = q_0^{\mathrm{II}} + q^{\mathrm{II}} \tag{42}$$

and s_1^{ii} and s_2^{ii} are the poles of $1/M(1/\widetilde{G}_{ii})$.

$$s_1^{ii} = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad s_2^{ii} = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \tag{43}$$

$$\begin{aligned} a &= A^{\mathrm{I}}V^{\mathrm{II}}\overline{G}_{ii}^{\mathrm{I}} + A^{\mathrm{II}}V^{\mathrm{I}}\overline{G}_{ii}^{\mathrm{II}} \\ b &= (p^{\mathrm{II}}q_0^{\mathrm{II}} + p^{\mathrm{I}}A^{\mathrm{II}})V^{\mathrm{I}}\overline{G}_{ii}^{\mathrm{II}} + (p^{\mathrm{I}}q_0^{\mathrm{I}} + p^{\mathrm{II}}A^{\mathrm{I}})V^{\mathrm{II}}\overline{G}_{ii}^{\mathrm{I}} \\ c &= p^{\mathrm{I}}p^{\mathrm{II}}(q_0^{\mathrm{II}}V^{\mathrm{I}}\overline{G}_{ii}^{\mathrm{II}} + q_0^{\mathrm{I}}V^{\mathrm{II}}\overline{G}_{ii}^{\mathrm{I}}) \end{aligned} \tag{44}$$

According to Eq. (39), inverse Laplace transformation of (40) is expressed as

$$\mathrm{Rev}\left(\frac{1}{M(1/\widetilde{G}_{ii})}\right) = \mathop{\mathrm{Res}}_{s=0}\left(\frac{\mathrm{P_{ii}(s)}}{\mathrm{s(s-s_1^{ii})(s-s_2^{ii})}}\mathrm{e^{st}}\right) + \mathop{\mathrm{Res}}_{s=s_1^{ii}}\left(\frac{\mathrm{P_{ii}(s)}}{\mathrm{s(s-s_1^{ii})(s-s_2^{ii})}}\mathrm{e^{st}}\right) + \mathop{\mathrm{Res}}_{s=s_2^{ii}}\left(\frac{\mathrm{P_{ii}(s)}}{\mathrm{s(s-s_1^{ii})(s-s_2^{ii})}}\mathrm{e^{st}}\right) \tag{45}$$

where, $\mathrm{Res}_{s=s_1}(f(s))$ denotes the residue of $f(s)$ in s_1. Thus, the relaxation modulus G_{ii}^H can be expressed in the following analytical formula

$$G_{ii}^H = \overline{G}_{ii}^{\mathrm{I}}\overline{G}_{ii}^{\mathrm{II}}\left[\frac{p^{\mathrm{I}}q_0^{\mathrm{I}}p^{\mathrm{II}}q_0^{\mathrm{II}}}{s_1s_2} + \frac{(A^{\mathrm{I}}s_1 + p^{\mathrm{I}}q_0^{\mathrm{I}})(A^{\mathrm{II}}s_1 + p^{\mathrm{II}}q_0^{\mathrm{II}})}{s_1(s_1-s_2)}\mathrm{e}^{s_1t} + \frac{(A^{\mathrm{I}}s_2 + p^{\mathrm{I}}q_0^{\mathrm{I}})(A^{\mathrm{II}}s_2 + p^{\mathrm{II}}q_0^{\mathrm{II}})}{(s_2-s_1)s_2}\mathrm{e}^{s_2t}\right]$$
$$i = 3,4,5 \tag{46}$$

From Eq. (37) and Eq. (46), the relaxation muduli can be computed analytically.

4.2. Model II

In the second model, every individual layer is assumed to be isotropic and its bulk deformation elastic, and the shear deformation satisfies the three-parameter solid model. The constitutive equation of the layers is expressed as

$$\{\sigma\} = \int_0^t \mathbf{G}(t-\xi)\frac{\partial\{\varepsilon(\xi)\}}{\partial\xi}\,\mathrm{d}\xi \tag{47}$$

Non-zero elements of the relaxation modulus matrix are

$$\mathbf{G}_{11} = \mathbf{G}_{22} = \mathbf{G}_{33} = K + \frac{2}{3}Y, \quad \mathbf{G}_{44} = \mathbf{G}_{55} = \mathbf{G}_{66} = Y, \quad \mathbf{G}_{12} = \mathbf{G}_{13} = \mathbf{G}_{23} = K - \frac{1}{3}Y \tag{48}$$

where K is the bulk elastic modulus and is not related to time, $Y(t)$ is the shear relaxation modulus and is expressed as

$$Y(t) = q_0 + q\mathrm{e}^{-pt}, \quad q_0 = 2G_1 - \frac{G_1^2}{G_1+G_2}, \quad q = \frac{G_1^2}{G_1+G_2}, \quad p = \frac{G_2}{\eta_2} \tag{49}$$

where G_1 and G_2 are shear rigidity moduli, and η_2 is viscous coefficient. In this case, the Laplace transformation of the effective relaxation modulus can be determined by Eq. (32). Considering the isotropic feature of the layers, and applying the inverse Laplace transformation to Eq. (32), the macroscopic effective relaxation modulus is expressed as

$$G_{33}^H = \mathrm{Rev}(1/M(1/\widetilde{G}_{33})) \tag{50a}$$

$$G_{11}^H = G_{22}^H = M(G_{11}) - M(\mathrm{Rev}(\widetilde{G}_{13}^2/(\widetilde{G}_{13}^2/\widetilde{G}_{33}))) + \mathrm{Rev}\left(\frac{M(\widetilde{G}_{13}/\widetilde{G}_{33})M(\widetilde{G}_{23}/\widetilde{G}_{33})}{M(1/\widetilde{G}_{33})}\right) \tag{50b}$$

$$G_{12}^H = M(G_{12}) - M(\mathrm{Rev}(\widetilde{G}_{13}^2/\widetilde{G}_{33})) + \mathrm{Rev}\left(\frac{M(\widetilde{G}_{13}/\widetilde{G}_{33})M(\widetilde{G}_{23}/\widetilde{G}_{33})}{M(1/\widetilde{G}_{33})}\right) \tag{50c}$$

$$G_{13}^H = G_{23}^H = \mathrm{Rev}\left(\frac{M(\widetilde{G}_{13}/\widetilde{G}_{33})}{M(1/\widetilde{G}_{33})}\right) \tag{50d}$$

$$G_{44}^H = G_{55}^H = \mathrm{Rev}(1/M(1/\widetilde{G}_{44})) \tag{50e}$$

$$\widetilde{G}_{66}^H = M(\widetilde{G}_{66}) \tag{50f}$$

The inverse Laplace transformations listed in the above equations will be determined in the following sections.

4.2.1. G_{33}^H and G_{44}^H

Firstly, we will derive the formulae to determine G_{33}^H and G_{44}^H. Using the method for deriving Eq. (40), we can obtain

$$1/M(1/\widetilde{G}_{33}) = \frac{1}{V^{\mathrm{I}}/\widetilde{G}_{33}^{\mathrm{I}} + V^{\mathrm{II}}/\widetilde{G}_{33}^{\mathrm{II}}} = \frac{P^{33}(s)}{3a^{33}s(s-s_1^{33})(s-s_2^{33})} \tag{51}$$

$$1/M(1/\widetilde{G}_{44}) = \frac{1}{V^{\mathrm{I}}/\widetilde{G}_{44}^{\mathrm{I}} + V^{\mathrm{II}}/\widetilde{G}_{44}^{\mathrm{II}}} = \frac{P^{44}(s)}{3a^{44}s(s-s_1^{44})(s-s_2^{44})} \tag{52}$$

where

$$a^{33} = V^{\mathrm{I}}(3K^{\mathrm{II}} + 2q_0^{\mathrm{II}} + 2q^{\mathrm{II}}) + V^{\mathrm{II}}(3K^{\mathrm{I}} + 2q_0^{\mathrm{I}} + 2q^{\mathrm{I}}) \tag{53a}$$

$$P^{33}(s) = \lfloor(3K^{\mathrm{I}} + 2q_0^{\mathrm{I}})(s+p^{\mathrm{I}}) + 2q^{\mathrm{I}}s\rfloor \cdot \lfloor(3K^{\mathrm{II}} + 2q_0^{\mathrm{II}})(s+p^{II}) + 2q^{\mathrm{II}}s\rfloor \tag{53b}$$

$$a^{44} = V^{\mathrm{I}}(q^{\mathrm{II}} + q_0^{\mathrm{II}}) + V^{\mathrm{II}}(q^{\mathrm{I}} + q_0^{\mathrm{I}}) \tag{54a}$$

$$P^{44}(s) = \lfloor(q^{\mathrm{I}} + q_0^{\mathrm{I}})s + q_0^{\mathrm{I}}p^{\mathrm{I}}\rfloor\lfloor(q^{\mathrm{II}} + q_0^{\mathrm{II}})s + q_0^{\mathrm{II}}p^{\mathrm{II}}\rfloor \tag{54b}$$

where s_1^{33} and s_2^{33} are the roots of the equation below

$$\begin{aligned} &V^{\mathrm{I}}(s+p^{\mathrm{I}})\lfloor s(3K^{\mathrm{II}} + 2q_0^{\mathrm{II}} + 2q^{\mathrm{II}}) + (3K^{\mathrm{II}} + 2q_0^{\mathrm{II}})p^{\mathrm{II}}\rfloor \\ &\quad + V^{\mathrm{II}}(s+p^{\mathrm{II}})[s(3K^{\mathrm{I}} + 2q_0^{\mathrm{I}} + 2q^{\mathrm{I}}) + (3K^{\mathrm{I}} + 2q_0^{\mathrm{I}})p^{\mathrm{I}}] = 0 \end{aligned} \tag{55}$$

and s_1^{44} and s_2^{44} are the roots of the below equation

$$V^{\mathrm{I}}(s+p^{\mathrm{I}})\lfloor(q^{\mathrm{II}} + q_0^{\mathrm{II}})s + q_0^{\mathrm{II}}p^{\mathrm{II}}\rfloor + V^{\mathrm{II}}(s+p^{\mathrm{II}})\lfloor(q^{\mathrm{I}} + q_0^{\mathrm{I}})s + q_0^{\mathrm{I}}p^{\mathrm{I}}\rfloor = 0 \tag{56}$$

Using the method expressed by Eq. (39), it will be derived that

$$G_{33}^H = \frac{P^{33}(0)}{3a^{33}s_1^{33}s_2^{33}} + \frac{P^{33}(s_1^{33})}{3a^{33}s_1^{33}(s_1^{33}-s_2^{33})}\mathrm{e}^{s_1^{33}1t} + \frac{P^{33}(s_2^{33})}{3a^{33}(s_2^{33}-s_1^{33})s_2^{33}}\mathrm{e}^{s_2^{33}t} \tag{57}$$

$$G_{44}^H = \frac{P^{44}(0)}{a^{44}s_1^{44}s_2^{44}} + \frac{P^{44}(s_1^{44})}{a^{44}s_1^{44}(s_1^{44}-s_2^{44})}\mathrm{e}^{s_1^{44}1t} + \frac{P^{44}(s_2^{44})}{a^{44}(s_2^{44}-s_1^{44})s_2^{44}}\mathrm{e}^{s_2^{44}t} \tag{58}$$

4.2.2. Inverse Laplace transformation of $\widetilde{G}_{13}^2/\widetilde{G}_{33}$

From Eqs. (48) and (49), we have

$$\widetilde{G}_{13}^2/\widetilde{G}_{33} = \frac{((3K-q_0)(s+p)-qs)^2}{3(3K+2q_0+2q)\left(s+\frac{(3K+2q_0)p}{3K+2q_0+2q}\right)} \cdot \frac{1}{(s+p)s} \tag{59}$$

By use of Eq. (39), we have

$$\mathrm{Rev}(\widetilde{G}_{13}^2/\widetilde{G}_{33}) = \frac{((3K-q_0)(s_1+p)-qs_1)^2}{3(3K+2q_0+2q)}\cdot\frac{1}{(s_1+p)s_1}\mathrm{e}^{s_1 t}+\frac{(3K-q_0)^2}{3(3K+2q_0)} - \frac{q^2 p}{3(3K+2q_0+2q)(s_1-p)}\mathrm{e}^{-pt} \tag{60}$$

where

$$s_1 = -\frac{(3K+2q_0)p}{3K+2q_0+2q} \tag{61}$$

4.2.3. Inverse Laplace transformation of $M(\widetilde{G}_{13}/\widetilde{G}_{33})/M(1/\widetilde{G}_{33})$

Defining

$$Q^{13}(s) = V^{\mathrm{I}}\lfloor(3K^{\mathrm{I}}-q_0^{\mathrm{I}}-q^{\mathrm{I}})s+(3K^{\mathrm{I}}-q_0^{\mathrm{I}})p^{\mathrm{I}}\rfloor\lfloor(3K^{\mathrm{II}}+2q_0^{\mathrm{II}}+2q^{\mathrm{II}})s+(3K^{\mathrm{II}}+2q_0^{\mathrm{II}})p^{\mathrm{II}}\rfloor + V^{\mathrm{II}}[(3K^{\mathrm{II}}-q_0^{\mathrm{II}}-q^{\mathrm{II}})s+(3K^{\mathrm{II}}-q_0^{\mathrm{II}})p^{\mathrm{II}}][(3K^{\mathrm{I}}+2q_0^{\mathrm{I}}+2q^{\mathrm{I}})s+(3K^{\mathrm{I}}+2q_0^{\mathrm{I}})p^{\mathrm{I}}] \tag{62}$$

then

$$M(\widetilde{G}_{13}/\widetilde{G}_{33}) = Q^{13}(s)/P^{33}(s) \tag{63}$$

and

$$\frac{M(\widetilde{G}_{13}/\widetilde{G}_{33})}{M(1/\widetilde{G}_{33})} = \frac{Q^{13}(s)}{P^{ss}(s)}\frac{1}{M(1/\widetilde{G}_{33})} = \frac{Q^{13}(s)}{3a^{33}s(s-s_1^{33})(s-s_2^{33})} \tag{64}$$

Based on Eq. (39), it is derived that

$$\mathrm{Rev}\left(\frac{M(\widetilde{G}_{13}/\widetilde{G}_{33})}{M(1/\widetilde{G}_{33})}\right) = \frac{Q^{13}(0)}{3a^{33}s_1^{33}s_2^{33}}+\frac{Q^{13}(s_1^{33})}{3a^{33}s_1^{33}(s_1^{33}-s_2^{33})}\mathrm{e}^{ts_1^{33}}+\frac{Q^{13}(s_2^{33})}{3a^{33}s_2^{33}(s_2^{33}-s_1^{33})}\mathrm{e}^{ts_2^{33}} \tag{65}$$

4.2.4. Inverse Laplace transformation of $M(\widetilde{G}_{13}/\widetilde{G}_{33})M(\widetilde{G}_{23}/\widetilde{G}_{33})/M(1/\widetilde{G}_{33})$

From Eq. (63), we have

$$\frac{M(\widetilde{G}_{13}/\widetilde{G}_{33})M(\widetilde{G}_{23}/\widetilde{G}_{33})}{M(1/\widetilde{G}_{33})} = \frac{[Q^{13}(s)]^2}{3a^{33}s(s-s_1^{33})(s-s_2^{33})P^{ss}(s)} = \frac{[Q^{13}(s)]^2}{3a^{33}s(s-s_1^{33})(s-s_2^{33})(s-s_3^{33})(s-s_4^{33})} \tag{66}$$

where s_3^{33} and s_4^{33} are the zero point of $P^{33}(s)$.

Table 1
Material constants of two different layers

	E (MPa)	υ	η (pad)	K (MPa)	q_0 (MPa)	q_1 (MPa)	p (1/day)
Mat-I	9.8×10^8	0.24	9.8×10^{10}	628.2	9.67×10^8	3.22×10^8	0.00658
Mat-II	1.96×10^8	0.2	9.8×10^{10}	108.9	6.1×10^7	1.84×10^8	0.00125

By using the residue theorem based method expressed by Eq. (39), the following formula is derived

$$\begin{aligned}\mathrm{Rev}\left[\frac{M(\widetilde{G}_{13}/\widetilde{G}_{33})M(\widetilde{G}_{23}/\widetilde{G}_{33})}{M(1/\widetilde{G}_{33})}\right] &= \frac{[Q^{13}(0)]^2}{3a^{33}s_1^{33}s_2^{33}s_3^{33}s_4^{33}} + \frac{[Q^{13}(s_1^{33})]^2}{3a^{33}s_1^{33}(s_1^{33}-s_2^{33})(s_1^{33}-s_3^{33})(s_1^{33}-s_4^{33})}\mathrm{e}^{s_1^{33}t} \\ &+ \frac{[Q^{13}(s_2^{33})]^2}{3a^{33}s_2^{33}(s_2^{33}-s_1^{33})(s_2^{33}-s_3^{33})(s_2^{33}-s_4^{33})}\mathrm{e}^{s_2^{33}t} \\ &+ \frac{[Q^{13}(s_3^{33})]^2}{3a^{33}s_3^{33}(s_3^{33}-s_1^{33})(s_3^{33}-s_2^{33})(s_3^{33}-s_4^{33})}\mathrm{e}^{s_3^{33}t} \\ &+ \frac{[Q^{13}(s_4^{33})]^2}{3a^{33}s_4^{33}(s_4^{33}-s_1^{33})(s_4^{33}-s_2^{33})(s_4^{33}-s_3^{33})}\mathrm{e}^{s_4^{33}t}\end{aligned} \tag{67}$$

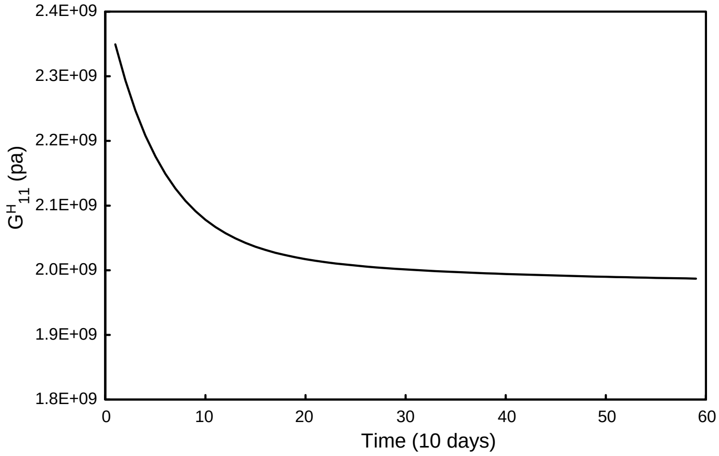

Fig. 2. Effective relaxation modulus $G_{11}^H (G_{22}^H = G_{11}^H)$ in time domain (the volume fraction of Mat-II is 10%).

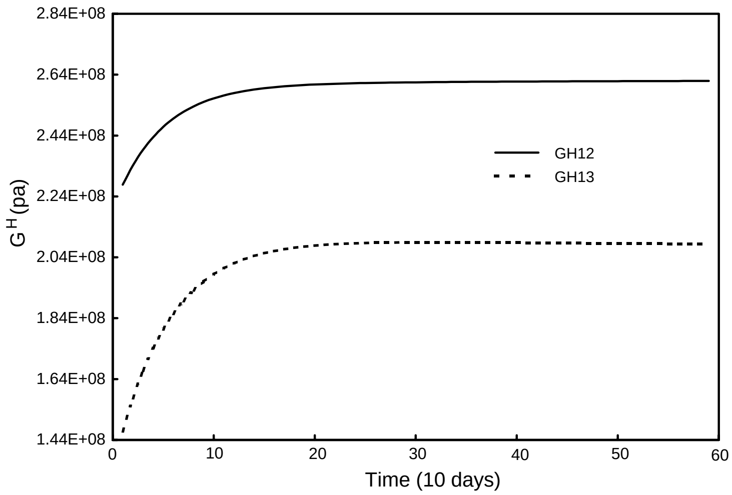

Fig. 3. Effective relaxation modulus G_{12}^H and G_{13}^H in time domain (the volume fraction of Mat-II is 10%).

After obtaining these inverse Laplace transformations, we can get every element of the relaxation modulus matrix of the layered material by substituting Eqs. (57), (58), (59), (61), (65) and (67) into Eq. (50).

5. Numerical example

Suppose that the microstructure of the layered material consists of two isotropic layers, and that the bulk deformation of every individual layer is elastic and the shear deformation satisfies the three-parameter solid model. The material constants of both layers are shown in Table 1. Based on the analytical expressions derived in this paper, the relaxation moduli of the layered materials are computed. Figs. 2–4 show the effective relaxation modulus in the time domain, when the volume fraction of MAT-II is 10%. Fig. 5 shows the variations of the effective relaxation modulus with the volume fraction of the layers.

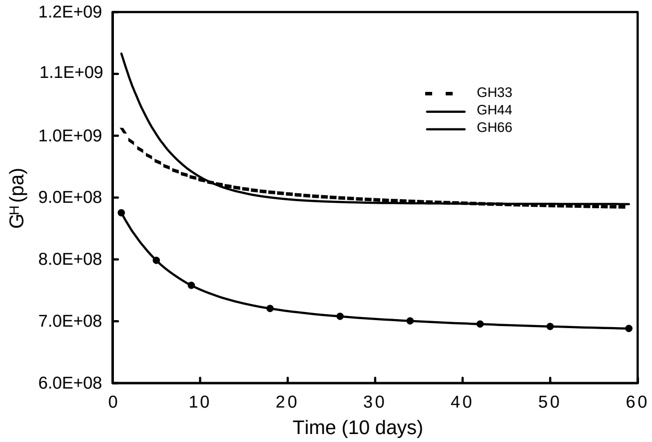

Fig. 4. Effective relaxation modulus G_{33}^H, G_{44}^H, G_{66}^H in time domain (the volume fraction of Mat-II is 10%).

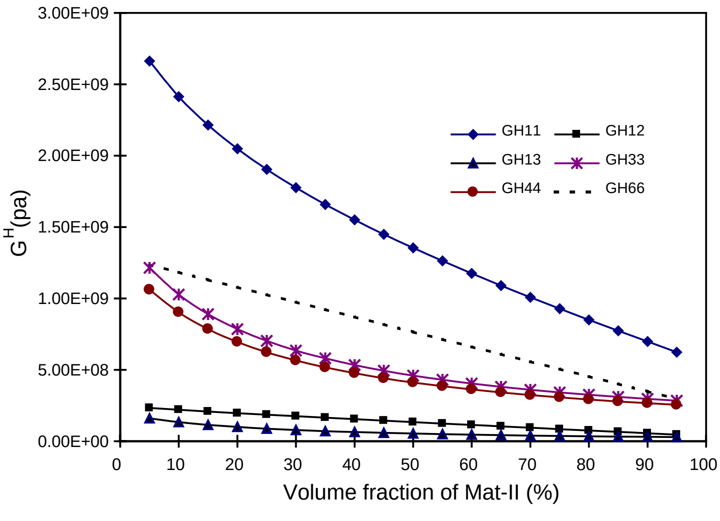

Fig. 5. Effective relaxation modulus vs. volume fraction of Mat-II (at the time of one day).

6. Conclusions

Based on the homogenization theory and Laplace transformation, analytical expressions for predicting the viscoelastic property of layered materials have been derived. The expressions make it easy to determine the overall viscoelastic property of layered materials and can be used in engineering design.

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